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Some Remarks About Flows in Hybrid Systems

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Abstract. We consider hybrid systems as networks consisting of continuous input-output systems and discrete input-output automata. Some of the outputs may be connected to some of the inputs; the others server as the inputs and outputs of the hybrid system. We define a class of regular flows for such systems and make some remarks about them.

1 Introduction

In this paper, a hybrid system is a network consisting of continuous input-output systems and discrete input-output automata. Some of the outputs may be connected to some of the inputs; the others serve as the inputs and outputs of the hybrid system. We are interested in flows of hybrid systems: to completely characterize flows is too difficult a problem. Indeed, this is a generalization of the problem of characterizing the flows of dynamical systems, which is already too hard. In this paper, we first show how the characterization of flows may be reduced to an algebraic problem and then make some remarks about this problem.

The purpose of this paper is to explain these ideas simply and to give some examples. Another more technical paper is in preparation which assumes a certain amount of background material in algebra, but gives the full definitions and provides the proofs for the concepts explained here [8]. It is not hard to implement systems which simulate and analyze hybrid systems of the type described here: a proof of concept implementation is described in [4].

There are a variety of interpretations for hybrid systems currently being explored. We mention three closely connected to the point of view in this paper. An automaton may be viewed as enabling or disabling some of the continuous input-output systems on the basis of discrete input symbols [7] and [6]. In other words, the hybrid system reflects some type of generalized mode switching. Alternatively, the automaton may be viewed as selecting trajectories or collections of trajectories of the continous systems in order to satisfy performance specifications [11]. Yet another interpretation is for automaton to be used to construct control laws for continous systems [10]. In this paper, we view hybrid systems from the first point of view.

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Hybrid systems have a variety of representations. The simplest are the state space representation and the input-output representation. In the state space representation, the states, inputs, and outputs of each component continuous system and automaton are described, as well as the input-output connections between the various systems. In the input-output representation, the inputs and outputs of the hybrid system as a whole are described.

In this paper, we use a different representation—the observation space representation. Roughly speaking, this may be viewed as dual to the state space representation. This representation is a very basic one: it forms the basis for the Heisenberg picture in quantum mechanics [3]; it has been used to define discrete time control systems by Sontag [13] and continuous time control systems by Bartosiewicz [1] and [2].

Using observation space representation, we define hybrid flows and regular hybrid flows. Regular hybrid flows are flows which are finite concatenations of flows of one or more continuous systems and mode switches. The important point is that at most a finite number of mode switches are involved. We first give an example of a hybrid flow which is not regular. We also give an an example of a hybrid system with the property that every point in a neighborhood of the origin can be reached using regular hybrid flows involving precisly one mode switch. Without mode switching, not every point in a nighborhood of the origin can be reached. These represent two extreme behaviors possible. It is an open problem to provide more precise characterizations of hybrid flows.

To work out the basic properties of hybrid flows from this viewpoint requires a certain amount of algebra. This is done in [8], with the results summarized here. In Section 2, we describe the observation space representation of input-output systems, automata, and hybrid systems, following [7]. In Section 3, we define hybrid flows and give the examples mentioned above.

2 Observation space representations

A basic principle is that the states X of a system can be recovered from the algebra of functions on the states $R = \mathrm{Fun}(X)$. R is one of several *observation algebras* that can be associated with a system. This leads to the observation space representation of a system which, broadly speaking, may be thought of as dual to the state space representation. To make this precise we define both a product structure on the space of observation functions, as is usual, and a dual structure, called a coproduct. This will allow us to view hybrid systems as suitable products of discrete automata and continuous control systems. As a by product of this approach, we can obtain as special cases the approach used in [13] to study discrete time control systems and the approach used in [1] and [2] to study continous time control systems.

The basic idea is that the time evolution of a state by the dynamics may be viewed as an action on the states, and that this action corresponds to an action on the algebra R. In the case of continuous systems, this action is derivation of

R; in the case of discrete state systems, this action is an endomorphism of R. We now explain this in more detail.

For continuous systems, a tangent vector to the space of states gives rise to a derivation E of the algebra R, that is, a linear map from R to itself satisfying

$$E(fg) = fE(g) + E(f)g, \qquad f, g \in R. \tag{1}$$

For discrete state systems, the action of the space of input words $w \in W$ on the states

$$X \times W \longrightarrow X, \qquad (x, w) \longmapsto x' = x \cdot w$$

induces an action on the observation algebra R

$$W \times R \longrightarrow R, \qquad (w, f) \longmapsto (w \cdot f)(x) = f'(x) = f(x \cdot w).$$

It is easy to see that the map

$$\sigma_w : R \longrightarrow R, \qquad \sigma_w(f) = f'$$

is an endomorphism of R, that is, a linear map from R to itself satisfying

$$\sigma(fg) = \sigma(f)\sigma(g), \qquad f, g \in R. \tag{2}$$

There is a natural generalization of these concepts which includes both of them: that is, a bialgebra H which acts on the algebra R in such a manner that

$$h(fg) = \sum_{(h)} h_{(1)}(f)h_{(2)}(g),$$

where $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$. We will see below how to define the coproduct Δ to obtain Equations 1 and 2.

Recall that a *coalgebra* is a vector space C with linear maps

$$\Delta : C \longrightarrow C \otimes C$$

(the coproduct) and

$$\epsilon : C \longrightarrow k$$

(the counit), satisfying conditions stating that Δ is coassociative, and that ϵ is a counit. We use the notation

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)},$$

to indicate that Δ sends an element c to some sum of terms of the form $c_{(1)} \otimes c_{(2)}$.

Coalgebras arise here since they are natural structures for describing actions on algebras. In particular, we will use them to describe the action on observation algebras of hybrid systems. We will require that

$$c(fg) = \Delta(c)(f \otimes g),$$

where $c \in C$ and $f, g \in R$, when C is a coalgebra acting on the algebra R. To say that c acts as a derivation is to say that c is *primitive*:

$$\Delta(c) = 1 \otimes c + c \otimes 1.$$

With this coproduct, we recover Equation 1. To say that c acts as an algebra endomorphism is to say that c is *grouplike*:

$$\Delta(c) = c \otimes c.$$

With this coproduct, we recover Equation 2. Denote the set of grouplikes of the coalgebra C by $\Gamma(C)$.

A *bialgebra* is an algebra and a coalgebra, in which the coalgebra maps are algebra homomorphisms. In the most general terms, a hybrid system in an *observation space* representation consists of a bialgebra H, a commutative algebra R with identity, and an action of H on R which satisfies

$$h(fg) = \sum_{(h)} h_{(1)}(f)h_{(2)}(g), \qquad \text{for } h \in H, \text{ and } f, g \in R.$$

That is, H acts on R with the primitives of H acting as derivations of R, and the grouplikes of H acting as endomorphisms of R. It is therefore a natural generalization of both continuous systems (modeled by derivations) and discrete state systems (modeled by discrete state transitions and the associated endomorphisms). Furthermore, as we shall see, it allows for hybrid systems exhibiting both continous and discrete behavior to built from discrete and continous components.

We now give a simple example of a hybrid system, following [7]. Continuous control systems can be viewed as special cases of this approach [5], as can discrete automata [7].

Example 1. In this example, we view a "taxicab on the streets of Manhattan" as a hybrid system with two modes: control in the first mode, corresponding to State 1, results east-west motion, but no north-south motion; control in the second mode, corresponding to State 2, results in north-south motion, but no east-west motion. See Figure 1. There are many ways of viewing this example. Of course, this system, due to its simplicity, can be modeled by a single control system in the plane consisting of a north-south vector field and an east-west vector field. The approach we describe here, on the other hand, generalizes to a large class of hybrid systems, viewed as mode switching of continous control systems, controlled by a discrete automaton. It is important to note that the automaton switches between two planar control systems, although the dynamics of each are in fact are constrained to one dimension in this case.

We begin by defining the state space representation of the hybrid system. Consider two control systems in the space k^2 of the form

$$\dot{x}(t) = u_1(t)E_1^{(i)}(x(t)) + u_2(t)E_2^{(i)}(x(t)), \quad \text{for } i = 1, 2,$$

where

$$E_1^{(1)} = \partial/\partial X_1, \quad E_2^{(1)} = 0,$$

and

$$E_1^{(2)} = 0, \quad E_2^{(2)} = \partial/\partial X_2.$$

A two state automaton accepts an input symbol, changes states, outputs a symbol, and on the basis of the output symbol selects a nonlinear system and a corresponding flow. See Figure 1 again.

To define the observation space representation of the system, let $R_i = k[X_1, X_2]$, $i = 1, 2$ denote the polynomial algebra on the indicated indeterminates and let $R = R_1 \oplus R_2$. Also, let $k<\xi_1, \xi_2>$ denote the free noncommutative associative algebra on the indicated indeterminates. We specify the action of the bialgebra $k<\xi_1, \xi_2>$ on R by specifying its actions on R_i, $i = 1, 2$: on R_1

$$\xi_1 \text{ acts as } E_1^{(1)} = \partial/\partial X_1,$$
$$\xi_2 \text{ acts as } E_2^{(1)} = 0;$$

on R_2

$$\xi_1 \text{ acts as } E_1^{(2)} = 0,$$
$$\xi_2 \text{ acts as } E_2^{(2)} = \partial/\partial X_2.$$

Consider input symbols a_1, corresponding to east-west travel only, and a_2, corresponding to north-south travel only. Let $G = \Omega^*$ be the semigroup (that is, the input words) freely generated by the input symbols $\Omega = \{a_1, a_2\}$. The action of Ω (and thus of G) on R is given by specifying its action on R_i, $i = 1$, 2. Its action on R_1 is given as follows: let $\rho_{12} : R_1 \to R_2$ be the isomorphism sending $X_1 \in R_1$ to $X_1 \in R_2$, and $X_2 \in R_1$ to $X_2 \in R_2$. Then, for $f \in R_1$,

$$a_i \cdot f = \begin{cases} f \oplus \rho_{12}(f) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Its action on R_2 is defined similarly. Intuitively, a_1 maps all states into State 1, and a_2 maps all states into State 2. The action of Ω on R is the transpose of this action. For simplicity, assume that the output symbol is given by the current state. With this assumption, the "typical" element $(u_1\xi_1 + u_2\xi_2)a_2(v_1\xi_1 + v_2\xi_2)a_1 \in k<\xi_1, \xi_2> \amalg k\Omega^*$ is to be interpreted as making a transition to State 1, flowing along $v_1 E_1^{(1)} + v_2 E_2^{(1)}$, making a transition to State 2, and then flowing along $u_1 E_1^{(2)} + u_2 E_2^{(2)}$. See Figure 2.

3 Flows

In this section, we consider flows of hybrid systems in the observation space representation. For technical reasons, we do not consider the most general type of hybrid system, but rather restrict attention to a smaller class. This restricted

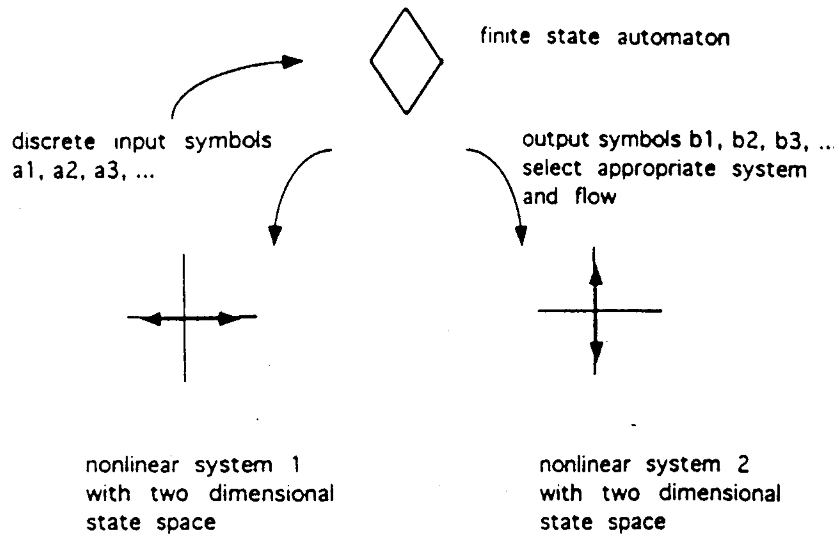

Fig. 1. The Manhattan taxicab hybrid system.

In this simple example, a finite state automaton with two states s_1 and s_2, accepts input symbols $a_1, a_2, a_3, \ldots$ transits states, and outputs a symbol $b_1, b_2, b_3, \ldots$ On the basis of the output symbol, a new nonlinear system and corresponding flow is selected. At the end of this flow, a new input symbol is accepted by the automaton and the cycle repeats.

class includes continuous systems, automata, and products of these, as in the example in the last section.

Define a *CDP bialgebra* (Continous and Discrete Product bialgebra) to be the free product of a primitively generated bialgebra H and a semigroup algebra G. Specifically, the free product is formed in the category of augmented algebras [12]. A bialgebra is called *primitively generated* if it is generated as an algebra by its primitive elements. This is the case for bialgebras corresponding to continuous systems [5], such as $H = k\langle\xi_1, \xi_2\rangle$ from Example 1. We consider a CDP bialgebra $H \amalg kG$ acting on an observation algebra R which is the direct sum of finitely many component algebras R_i which are associated with continuous systems (H, R_i). This is an immediate generalization of Example 1. The bialgebra $H \amalg kG$ acts on $R = \bigoplus_i R_i$ as follows. The bialgebra H acts on R_i as it does in the individual continuous systems (H, R_i); the semigroup G acts on the set of states $\{i\}$, and acts on the function algebra R in a fashion compatible with its action on the states. For more detail, see [7].

We discuss flows in the context of formal series. If V is a vector space, denote by V_t the space of formal power series $V[[t]] = \bigoplus_{n=0}^{\infty} Vt^n$ over V. In [9] completed tensor products of spaces of the form V_t and coalgebras of this form are discussed.

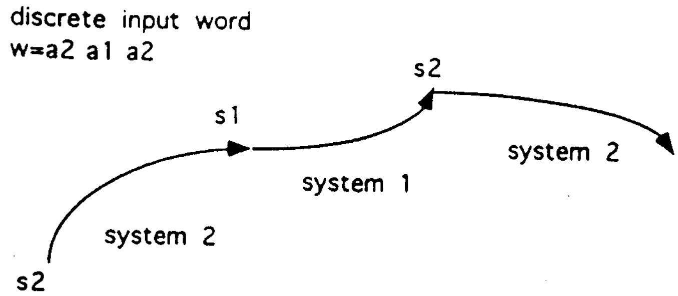

Fig. 2. Another view of the Manhattan taxicab hybrid system.

This is another view of the hybrid system illustrated in Figure 1. For this illustration, assume for simplicity that the input symbol i selects the corresponding state and nonlinear system. The flow sketched schematically above is the result of the input word $w = a_2 a_1 a_2$, which results in the state transition sequence $s_2 s_1 s_2$, which in turn selects the nonlinear systems indicated.

Formally, the solution to the differential equation

$$\dot{x}(t) = E(x(t)), \qquad x(0) = x^0$$

is given by

$$x(t) = e^{tE} x^0.$$

It can be shown that the fact that E is primitive implies that e^{tE} is grouplike. In the observation space representation of a continuous system, the grouplike $e^{tE} \in H_t$ is the flow corresponding to the differential equation $\dot{x}(t) = E(x(t))$. To summarize, the dynamics of a continuous systems are infinitesimally determined by derivations $E \in H$ while the flows are specified by grouplikes $K \in H_t$. One can think of K as being of the form $K = e^{tE}$.

In the observation space representation of the discrete automaton in which the alphabet of input symbols Ω acts on the state space of a finite automaton, the flows are exactly the grouplike elements of the bialgebra $(k\Omega^*)_t$, where Ω^* is the semigroup of words in the alphabet Ω. It can be shown that these elements are exactly the elements of Ω^*. To summarize, the dynamics of an automaton correspond to endomorphisms associated with input symbols, and the flows correspond to grouplike elements of $(k\Omega^*)_t$, which are words in the input symbols. The flows may be viewed as execution sequences of the automaton.

We turn to the general case now. Given any bialgebra B, an algebra of observation functions R, and a compatible action of B on R [7], the *flows* of the hybrid system are defined to be the grouplike elements of B_t. We pose the general problem:

Problem: Characterize the flows of hybrid systems.

As it stands, this is much too hard, since it includes as special cases the problems of characterizing the flows of dynamical systems, of automata, and of a large variety of systems formed from suitable products of these. In this note, we introduce a class of nicely behaved flows and make some comments about them.

In the observation space representation of the hybrid system in which the underlying bialgebra of the observation space representation is the CDP algebra $B = H \amalg kG$, the flows are the grouplikes in $(H \amalg kG)_t$. Note that the map from $H_t \amalg kG$ to $(H \amalg kG)_t$ induces a map from $\Gamma(H_t \amalg kG)$ to the flows $\Gamma((H \amalg kG)_t)$. The flows which are in the image of this map are ones which arise as a finite sequence of continuous flows (elements of $\Gamma(H_t)$) and mode switches (elements of $G = \Gamma(kG)$). We call these flows *regular flows.*

Example 2. Let $H \amalg kG$ dnote a CDP bialgebra. This example shows that not all flows are regular. Let E denote a primitive element of H, and let $g_1, g_2, \ldots$ be an infinite sequence of distinct invertible elements of G. Then

$$\prod_{n=1}^{\infty} g_n^{-1} e^{t^n E} g_n$$

is a flow in $\Gamma((H \amalg kG)_t)$ which is not in the image of $\Gamma(H_t) \amalg G$, since it cannot be expressed using only finitely many mode switches.

We observe next that the taxicab example described in Example 1 has the property that there exists a neighborhood of the origin of k^2 with the property that every point x' in the neighborhood is of the form $x' = Kx^0$, where x^0 is the origin and K is a regular flow. In other words, a neighborhood of the origin is reachable using regular flows. It is also easy to see that almost all these flows require at least one mode switch by an element of $G = \Omega^*$.

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References

1. Z. Bartosiewicz, "Ordinary differential equations on real affine varieties," *Bulletin of the Polish Academy of Sciences: Mathematics*, Volume 35, pp. 13-18, 1987.
2. Z. Bartosiewicz, "Minimal Polynomial Realizations," *Mathematics of Control, Signals, and Systems*, Volume 1, pp. 227-237, 1988.
3. J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, Springer-Verlag, New York, 1981.

4. R. L. Grossman, "Path planning, persistent stores, and hybrid systems," submitted for publication.
5. R. L. Grossman and R. G. Larson, "The realization of input–output maps using bialgebras, *Forum Math.*, Volume 4, pp. 109–121, 1992.
6. R. L. Grossman and R. G. Larson, "Viewing hybrid systems as products of control systems and automata," *Proceedings of the 31st IEEE Conference on Decision and Control*, IEEE Press, 1992.
7. R. L. Grossman and R. G. Larson, "A bialgebra approach to hybrid systems, submitted for publication.
8. R. L. Grossman and R. G. Larson, "Hybrid systems as free products of bialgebras, in preparation.
9. R. L. Grossman and D. E. Radford, "Bialgebra deformations of certain universal enveloping algebras," *Communications in Pure and Applied Algebra*, to appear.
10. W. Kohn, "Declarative control theory," *AIAA GN & D Journal*, February, 1989.
11. W. Kohn and A. Nerode, "Models for hybrid sysems: Automata, topologies and stability," submitted for publication.
12. M. E. Sweedler, *Hopf algebras*, W. A. Benjamin, New York, 1969.
13. E. D. Sontag, *Polynomial Response Maps*, Lecture Notes in Control and Information Sciences, Volume 13, Springer-Verlag, New York, 1979.